Structural Lawhood as Interior Admissibility: Phase Geometry of Operator Families Across Domains

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Abstract

We introduce a phase-theoretic framework for structural stability under admissible perturbations of operator families. A structural signature is said to exhibit lawhood when it persists as an invariant across a perturbation class whose magnitude is bounded relative to intrinsic spectral separation. We formalize this persistence via a rigidity modulus, establish a sharp boundary theorem separating stable and degeneracy-admissible regimes, and describe the resulting admissibility manifold as a stratified phase geometry. We prove a matching-number theorem characterizing degeneracy structure combinatorially, and we demonstrate empirical instantiation in a seismic ranking system. The same phase structure appears in independent geometric projection problems, indicating cross-domain universality.

1 Introduction

Many scientific systems exhibit structural patterns that appear stable under perturbation, truncation, or measurement noise, while others collapse under minimal deformation. A principled distinction between invariant structure and accidental coincidence is lacking.

We develop a phase-theoretic criterion for structural persistence under admissible perturbations of operator families. The resulting rigidity modulus defines a sharp phase boundary separating invariant and degeneracy-admissible regimes.

2 Admissible Operator Families and Structural Signatures

Let $\mathcal{X}$ be a finite-dimensional real or complex Hilbert space with norm $\|\cdot\|$ induced by inner product $\langle \cdot, \cdot \rangle$.

Let $\mathcal{O}$ denote a family of bounded linear operators

$$O : \mathcal{X} \to \mathcal{X}.$$

We assume each $O \in \mathcal{O}$ admits a spectral decomposition (e.g., is normal or diagonalisable over $\mathbb{C}$).

2.1 Structural Signatures

Each operator O induces a structural signature

$$S(O),$$

which may include:

- Ordered eigenvalues $\{\lambda_i(O)\}$,

- Spectral gaps $\lambda_i - \lambda_j$,

- Eigenvector directions up to phase,

- Induced rankings,

- Projection axes,

- Power ratios or quadratic functionals.

We assume $S(O)$ depends continuously on O under the chosen operator norm.

2.2 Perturbations

Let $O_0 \in \mathcal{O}$ be a reference operator.

Let $\| \cdot \|$ denote the operator norm induced by $\mathcal{X}$:

$$\|O\| = \sup_{\|x\|=1} \|Ox\|.$$

Definition 2.1 (Admissible Perturbation). A perturbation δO is admissible if:

$$\|\delta O\| \leq \epsilon,$$

for fixed $\epsilon > 0$, and $O_0 + \delta O \in \mathcal{O}$.

We define the admissible family:

$$\mathcal{O}_\epsilon(O_0) = \{O_0 + \delta O : \|\delta O\| \leq \epsilon\}.$$

2.3 Spectral Separation

Let $\{\lambda_i\}$ denote the eigenvalues of O_0, ordered so that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

Define the minimal spectral gap:

$$\Delta_{\min} = \min_{i \neq j} |\lambda_i - \lambda_j|.$$

We assume $\Delta_{\min} > 0$ for the reference operator.

2.4 Angular Separation

If the structural signature depends on eigenvector directions, let $\{v_i\}$ denote normalized eigenvectors.

Define minimal angular separation:

$$\Theta_{\min} = \min_{i \neq j} \arccos \left(|\langle v_i, v_j \rangle| \right).$$

2.5 Perturbation Envelope Bounds

Classical perturbation theory (e.g., Weyl inequalities) yields:

$$|\lambda_i(O_0 + \delta O) - \lambda_i(O_0)| \leq \|\delta O\|. \tag{1}$$

Thus the maximal eigenvalue shift under admissible perturbation satisfies:

$$\sigma_P = \epsilon.$$

For eigenvectors, Davis–Kahan-type bounds imply that angular deviation satisfies:

$$\delta_P \leq \frac{\|\delta O\|}{\Delta_{\min}},$$

provided $\Delta_{\min} > 0$.

2.6 Admissibility Criterion

The structural signature $S(O_0)$ is said to be *admissibly invariant* if:

$$S(O_0 + \delta O) = S(O_0) \quad \text{for all admissible } \delta O.$$

The question becomes quantitative:

Under what condition on $(\Delta_{\min}, \Theta_{\min}, \epsilon)$ does admissible invariance hold?

3 Rigidity Modulus and Phase Boundary

3.1 A general structural phase theorem (rigidity coordinate from bounded perturbations)

We isolate a minimal, framework-independent mechanism that forces a phase coordinate of the "distance-to-instability" type in any operator system that admits (i) a bounded perturbation envelope, (ii) a separation notion between competing structural outputs, and (iii) a continuous (Lipschitz) signature extraction.

Definition 3.1 (Operator family with bounded envelope). Let $(\mathcal{P}, d_{\mathcal{P}})$ be a metric space of operator parameters and let X be a data/representation space equipped with a metric d_X. An operator family is a map $\mathcal{O} : \mathcal{P} \to \mathrm{End}(X)$, $p \mapsto \mathcal{O}_p$. Fix a baseline parameter $p_0 \in \mathcal{P}$ and a radius $\varepsilon > 0$. The corresponding bounded perturbation envelope is the metric ball

$$B_\varepsilon(p_0) := \{p \in \mathcal{P} : d_{\mathcal{P}}(p, p_0) \leq \varepsilon\}.$$

Definition 3.2 (Signature map and operator-Lipschitz bound). Let (Y, d_Y) be a metric space of signatures and let $\Sigma : X \to Y$ be a signature extraction map. Define the induced map $F_{x_0} : \mathcal{P} \to Y$ by

$$F_{x_0}(p) := \Sigma(\mathcal{O}_p(x_0)),$$

where $x_0 \in X$ is the fixed baseline input (the observed object). Assume there exists a constant $L \geq 0$ such that for all $p, q \in B_\varepsilon(p_0)$,

$$d_Y\big(F_{x_0}(p), F_{x_0}(q)\big) \leq L \, d_{\mathcal{P}}(p, q).$$

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Definition 3.3 (Separation margin)**.** Let $\mathcal{C} \subset Y$ be a closed critical set representing ambiguity or degeneracy. Define the separation margin at the baseline as

$$\Delta(p_0) := \inf_{c \in \mathcal{C}} d_Y\big(F_{x_0}(p_0), c\big).$$

Definition 3.4 (Rigidity coordinate)**.** Define the dimensionless rigidity coordinate

$$R(p_0) := \frac{\Delta(p_0)}{2L\varepsilon},$$

with the convention that $R(p_0) = +\infty$ if $L = 0$ and $\Delta(p_0) > 0$.

Theorem 3.5 (Structural phase theorem: interior rigidity and boundary at $R = 1$)**.** *Assume* $\Delta(p_0) > 0$ *and that* F_{x_0} *is* L-*Lipschitz on* $B_\varepsilon(p_0)$. *Then:*

1. *(Interior rigidity) If* $R(p_0) > 1$, *then for every* $p \in B_\varepsilon(p_0)$,

$$d_Y\big(F_{x_0}(p), \mathcal{C}\big) \geq \Delta(p_0) - L\, d_\mathcal{P}(p, p_0) \geq \Delta(p_0) - L\varepsilon > 0.$$

2. *(Sharp boundary) The threshold* $R(p_0) = 1$ *is the natural phase boundary for admissible instability.*

3. *(Witness condition) If* $R(p_0) \leq 1$ *and there exists* $p^\star \in B_\varepsilon(p_0)$ *such that*

$$d_Y\big(F_{x_0}(p^\star), \mathcal{C}\big) \leq \Delta(p_0) - L\varepsilon,$$

then admissible instability is attainable within the envelope.

Proof. For any $p \in B_\varepsilon(p_0)$ and any $c \in \mathcal{C}$, the triangle inequality gives

$$d_Y\big(F_{x_0}(p), c\big) \geq d_Y\big(F_{x_0}(p_0), c\big) - d_Y\big(F_{x_0}(p), F_{x_0}(p_0)\big) \geq d_Y\big(F_{x_0}(p_0), c\big) - L\, d_\mathcal{P}(p, p_0).$$

Taking the infimum over $c \in \mathcal{C}$ yields

$$d_Y\big(F_{x_0}(p), \mathcal{C}\big) \geq \Delta(p_0) - L\, d_\mathcal{P}(p, p_0) \geq \Delta(p_0) - L\varepsilon.$$

If $R(p_0) > 1$, then $\Delta(p_0) - L\varepsilon > 0$. $\qquad\square$

This section derives a quantitative criterion guaranteeing invariance of structural signatures under admissible perturbations. We treat two common signature components: (i) spectral order (rankings) and (ii) directional structure (eigenspaces/axes).

3.2 Spectral-order stability

Let O_0 have real eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ (for normal/self-adjoint O_0; otherwise interpret λ_i as the ordered real spectrum of the relevant Hermitian form defining the signature). Let $\Delta_{\min}$ be the minimal eigenvalue separation:

$$\Delta_{\min} = \min_{i \neq j} |\lambda_i - \lambda_j| > 0.$$

Let $O = O_0 + \delta O$ with $\|\delta O\| \leq \epsilon$. By Weyl-type bounds,

$$\max_i |\lambda_i(O) - \lambda_i(O_0)| \leq \|\delta O\| \leq \epsilon. \tag{2}$$

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Lemma 3.6 (No-crossing under a gap margin). *If $2\epsilon < \Delta_{\min}$, then the ordering of the eigenvalues is preserved under any admissible perturbation: for all $i < j$,*

$$\lambda_i(O) > \lambda_j(O),$$

and in particular the ranking induced by $\{\lambda_i\}$ is invariant on $\mathcal{O}_\epsilon(O_0)$.

Proof. Fix $i < j$. By (2),

$$\lambda_i(O) \geq \lambda_i(O_0) - \epsilon, \qquad \lambda_j(O) \leq \lambda_j(O_0) + \epsilon.$$

Thus

$$\lambda_i(O) - \lambda_j(O) \geq (\lambda_i(O_0) - \lambda_j(O_0)) - 2\epsilon \geq \Delta_{\min} - 2\epsilon.$$

If $2\epsilon < \Delta_{\min}$, the right-hand side is positive, hence $\lambda_i(O) > \lambda_j(O)$ for all admissible O, and no eigenvalue crossing can occur. Therefore the induced order/ranking is invariant. $\square$

We summarize the spectral stability margin by defining the *spectral rigidity coordinate*

$$R_{\mathrm{spec}} := \frac{\Delta_{\min}}{2\epsilon}.$$

Remark. The factor 2 reflects symmetric allocation of perturbation budget; any constant $c > 1$ would produce an equivalent phase coordinate up to multiplicative rescaling. Lemma 3.6 states that $R_{\mathrm{spec}} > 1$ implies spectral-order invariance.

3.3 Directional (eigenspace/axis) stability

Many structural signatures depend not only on eigenvalues but also on eigenvectors or invariant subspaces (e.g., principal axes). Directional stability is controlled by spectral separation and perturbation size.

Let v be a unit eigenvector of O_0 associated to a simple eigenvalue λ separated from the rest of the spectrum by gap

$$\Delta(\lambda) := \min_{\mu \in \mathrm{spec}(O_0) \backslash \{\lambda\}} |\lambda - \mu|.$$

When the signature uses a distinguished axis (e.g., top eigendirection), we take $\Delta(\lambda)$ for that eigenvalue and set $\Delta_{\min}$ accordingly.

For normal/self-adjoint operators, Davis–Kahan-type bounds imply that the angle between the perturbed and unperturbed eigenvectors satisfies

$$\sin \angle(v, \tilde{v}) \leq \frac{\|\delta O\|}{\Delta(\lambda)} \leq \frac{\epsilon}{\Delta_{\min}}, \tag{3}$$

where $\tilde{v}$ denotes the corresponding perturbed eigenvector chosen with the usual phase/sign convention.

Now suppose the signature includes a *directional separation constraint*, e.g. an angle $\Theta_{\min}$ between two distinguished axes that must not be violated to preserve the structural relation (such as classification into distinct directional regimes, or a non-degeneracy condition of axis identification). Define $\Theta_{\min} > 0$ as the minimal angular margin required to maintain the directional component of the signature.

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Lemma 3.7 (Directional invariance under angular margin)**.** *Assume the directional component of the signature remains unchanged whenever each distinguished axis is perturbed by at most δ_P in angle, and that δ_P is bounded by (3). If*

$$2\delta_P < \Theta_{\min},$$

then the directional component of $S(O_0)$ is invariant under all admissible perturbations.

Proof. Under admissible perturbations, each distinguished axis rotates by at most δ_P. Therefore the relative angular configuration can change by at most $2\delta_P$ (worst-case in opposite directions). If $2\delta_P < \Theta_{\min}$, the configuration cannot cross the margin required to alter the directional component of the signature; hence it remains invariant for all admissible perturbations. $\square$

We summarize directional stability by defining the *directional rigidity coordinate*

$$R_{\mathrm{dir}} := \frac{\Theta_{\min}}{2\delta_P}.$$

Lemma 3.7 states that $R_{\mathrm{dir}} > 1$ implies directional invariance.

3.4 Rigidity modulus

Structural signatures often include both spectral-order and directional components. We therefore define a combined modulus.

Definition 3.8 (Rigidity modulus)**.** Let ϵ be the admissible perturbation envelope, let $\Delta_{\min}$ be the minimal spectral gap relevant to the signature, and let $\Theta_{\min}$ be the minimal directional margin relevant to the signature. Let δ_P be any bound satisfying (3) (or a domain-calibrated angular perturbation bound). Define the rigidity modulus:

$$R := \min\left(\frac{\Delta_{\min}}{2\epsilon}, \frac{\Theta_{\min}}{2\delta_P}\right).$$

3.5 Phase boundary theorem

Theorem 3.9 (Rigidity boundary)**.** *Let $S(O)$ be a structural signature whose stability decomposes into (i) a spectral-order component and (ii) a directional component, each preserved under the respective margin conditions of Lemma 3.6 and Lemma 3.7.*

If $R > 1$, then $S(O)$ is invariant on the admissible family $\mathcal{O}_\epsilon(O_0)$:

$$S(O_0 + \delta O) = S(O_0) \quad \text{for all } \|\delta O\| \leq \epsilon.$$

If $R \leq 1$, then at least one stability margin fails; consequently the admissible family intersects a regime where either spectral crossing or directional degeneracy is not excluded by perturbation bounds, and structural instability may occur.

Proof. If $R > 1$, then both coordinates exceed 1:

$$\frac{\Delta_{\min}}{2\epsilon} > 1 \quad \text{and} \quad \frac{\Theta_{\min}}{2\delta_P} > 1.$$

Equivalently, $2\epsilon < \Delta_{\min}$ and $2\delta_P < \Theta_{\min}$. By Lemma 3.6, spectral-order structure is invariant for all admissible perturbations. By Lemma 3.7, the directional component is also invariant for all admissible perturbations. Since S is the combination of these components, S is invariant on $\mathcal{O}_\epsilon(O_0)$.

If $R \leq 1$, then at least one of the inequalities fails: either $2\epsilon \geq \Delta_{\min}$ or $2\delta_P \geq \Theta_{\min}$. In the first case, eigenvalue crossing is not excluded by the perturbation envelope and therefore ranking instability cannot be ruled out. In the second case, directional configuration change beyond the margin is not excluded and directional instability cannot be ruled out. In either case, the admissible family intersects a regime where the corresponding component of S is not guaranteed to remain invariant, hence instability may occur. $\qquad\square$

Remark 3.10 (Interpretation of the phase boundary). The hypersurface $R = 1$ defines a sharp boundary between an interior regime (invariance guaranteed) and a boundary/degeneracy-admissible regime (invariance not guaranteed). Later sections interpret this boundary geometrically and empirically.

4 Stratified Admissibility Geometry

We now formalize the geometric structure induced by the rigidity modulus.

4.1 Parameter Space

Let $\mathcal{P}$ denote the parameter space governing the operator family. Elements $p \in \mathcal{P}$ determine:

$$O(p) \in \mathcal{O}, \quad \epsilon(p), \quad \delta_P(p).$$

We assume:

- $\mathcal{P}$ is a smooth finite-dimensional manifold,

- the map $p \mapsto O(p)$ is smooth in operator norm,

- $\epsilon(p)$ and $\delta_P(p)$ are continuous functions.

Define the spectral gap and directional separation functions:

$$\Delta_{\min}(p), \qquad \Theta_{\min}(p),$$

as in Section 2, computed for $O(p)$.

4.2 Admissibility Map

Define the admissibility map

$$\Phi : \mathcal{P} \longrightarrow \mathbb{R}^2$$

by

$$\Phi(p) = \left(\frac{\Delta_{\min}(p)}{2\epsilon(p)}, \; \frac{\Theta_{\min}(p)}{2\delta_P(p)} \right).$$

Define the rigidity modulus function

$$R(p) := \min\left(\Phi_1(p), \Phi_2(p) \right).$$

By construction, $R : \mathcal{P} \to \mathbb{R}_{\geq 0}$ is continuous.

4.3 Phase Regions

Define the following subsets of parameter space:

$$\mathcal{P}_{\text{int}} = \{p \in \mathcal{P} : R(p) > 1\},$$
$$\mathcal{P}_{\text{bdry}} = \{p \in \mathcal{P} : R(p) = 1\},$$
$$\mathcal{P}_{\text{deg}} = \{p \in \mathcal{P} : R(p) < 1\}.$$

Proposition 4.1 (Interior openness). *$\mathcal{P}_{\text{int}}$ is an open subset of $\mathcal{P}$.*

Proof. R is continuous as a minimum of continuous functions. Therefore $\mathcal{P}_{\text{int}} = R^{-1}((1,\infty))$ is open. $\square$

4.4 Boundary Regularity

Assume that at $p_0 \in \mathcal{P}_{\text{bdry}}$,

$$\nabla R(p_0) \neq 0.$$

Theorem 4.2 (Phase boundary regularity). *Under the nondegeneracy condition $\nabla R(p_0) \neq 0$, $\mathcal{P}_{\text{bdry}}$ is locally a codimension-one embedded submanifold of $\mathcal{P}$ near p_0.*

Proof. R is continuous and piecewise smooth away from degeneracy of Φ_1 and Φ_2. Near p_0, assume R equals one coordinate function Φ_i in a neighborhood. Then $R(p) = 1$ locally defines a level set of a smooth function with nonvanishing gradient. The implicit function theorem implies that this level set is a codimension-one embedded submanifold. $\square$

Thus the phase boundary generically forms a hypersurface.

4.5 Degeneracy Sets

Degeneracy corresponds to vanishing spectral gap or vanishing directional separation:

$$\mathcal{D}_{\text{spec}} = \{p : \Delta_{\min}(p) = 0\},$$

$$\mathcal{D}_{\text{dir}} = \{p : \Theta_{\min}(p) = 0\}.$$

These sets are algebraic (or analytic) degeneracy loci defined by eigenvalue coincidence or axis coincidence conditions.

They form lower-dimensional stratified subsets of $\mathcal{P}$.

4.6 Stratification Structure

The parameter space decomposes into strata determined by:

- inequalities $R(p) > 1$, $R(p) < 1$,

- equalities $R(p) = 1$,

- degeneracy constraints $\Delta_{\min}(p) = 0$ or $\Theta_{\min}(p) = 0$,

- combinatorial multiplicity of eigenvalue coincidences.

This induces a stratification of $\mathcal{P}$ into smooth manifolds of varying dimension, with boundary hypersurface separating interior invariant regime from degeneracy-admissible regime.

4.7 Geometric Interpretation

The admissibility geometry consists of:

- An open interior region $\mathcal{P}_{\text{int}}$ where structural signatures are invariant,

- A codimension-one boundary hypersurface $\mathcal{P}_{\text{bdry}}$,

- Lower-dimensional degeneracy strata where spectral or directional collapse occurs.

This defines a phase geometry over parameter space.

5 Degeneracy Matching Number Theorem

We now characterize spectral degeneracy combinatorially in terms of perturbation-induced order crossings.

5.1 Ranking Structure and Crossing Graph

Let O_0 have ordered eigenvalues

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n,$$

with minimal spectral gap $\Delta_{\min} > 0$.
Under admissible perturbation $\|\delta O\| \leq \epsilon$, eigenvalue shifts satisfy

$$|\lambda_i(O) - \lambda_i(O_0)| \leq \epsilon.$$

A pair (i, j) with $i < j$ is said to be *crossing-admissible* if

$$|\lambda_i(O_0) - \lambda_j(O_0)| \leq 2\epsilon.$$

This condition is necessary for order reversal to be possible under admissible perturbation.

Definition 5.1 (Crossing Graph)**.** Define a graph $G(O_0, \epsilon) = (V, E)$ where:

- $V = \{1, \ldots, n\}$,

- $(i, j) \in E$ if the pair (i, j) is crossing-admissible.

Thus edges represent pairs whose ordering is not protected by the perturbation envelope.

5.2 Matching Number

Definition 5.2 (Matching Number)**.** The matching number $\nu(G)$ is the maximum number of pairwise disjoint edges in G.

Intuitively, $\nu(G)$ counts the maximal number of independent eigenvalue pairs that can simultaneously undergo crossing under admissible perturbations.

5.3 Interior Characterization

Theorem 5.3 (Degeneracy Matching Theorem). *Let $R_{\mathrm{spec}} = \Delta_{\min}/(2\epsilon)$.*
Then:

$$R_{\mathrm{spec}} > 1 \quad \Longleftrightarrow \quad \nu(G) = 0.$$

Proof. $(\Rightarrow)$
If $R_{\mathrm{spec}} > 1$, then

$$\Delta_{\min} > 2\epsilon.$$

Thus for all $i \neq j$,

$$|\lambda_i - \lambda_j| > 2\epsilon.$$

Therefore no pair satisfies the crossing-admissibility condition. Hence $E = \emptyset$ and $\nu(G) = 0$.
$(\Leftarrow)$
If $\nu(G) = 0$, then $E = \emptyset$, since any edge would constitute a matching of size at least one.
Thus no pair satisfies

$$|\lambda_i - \lambda_j| \leq 2\epsilon.$$

Therefore

$$\Delta_{\min} > 2\epsilon,$$

which implies

$$R_{\mathrm{spec}} > 1.$$

$\square$

5.4 Beyond Binary Degeneracy

When $R_{\mathrm{spec}} \leq 1$, the crossing graph may contain multiple edges.
    The matching number $\nu(G)$ then quantifies degeneracy capacity:

- $\nu(G) = 1$: single independent crossing admissible,

- $\nu(G) = k$: k independent crossing pairs admissible,

- $\nu(G) = \lfloor n/2 \rfloor$: maximal combinatorial degeneracy.

    Thus $\nu(G)$ measures distance into the degeneracy-admissible region in discrete combinatorial units.

5.5 Relation to Rigidity Modulus

Recall that

$$R = \min(R_{\mathrm{spec}}, R_{\mathrm{dir}}).$$

    Interior admissibility requires $\nu(G) = 0$ and directional stability simultaneously.
    Thus spectral matching number and directional modulus jointly determine the full structural phase.

5.6 Stratified Combinatorial Structure

Within $\mathcal{P}_{\text{deg}}$, parameter space further decomposes according to matching number:

$$\mathcal{P}_k = \{p \in \mathcal{P} : \nu(G(p)) = k\}.$$

Each $\mathcal{P}_k$ defines a combinatorial stratum corresponding to k independent degeneracy channels. These strata refine the geometric stratification defined in Section 4.

6 Empirical Instantiation: Seismic Ranking Stability

We now instantiate the rigidity framework in a real-world ranking and directional clustering system.

6.1 Data and Operator Construction

Let an event consist of a finite set of observations

$$\mathcal{E} = \{x_1, \ldots, x_m\}.$$

Each observation x_i carries:

- A scalar magnitude M_i,

- A directional coordinate d_i in a metric space.

Define the magnitude operator

$$O_{\text{mag}} : \mathbb{R}^m \to \mathbb{R}^m,$$

whose spectrum consists of ordered magnitudes

$$\lambda_i = M_{(i)},$$

sorted in decreasing order.

Thus spectral gaps correspond to differences between adjacent ordered magnitudes.

Directional structure is represented by a clustering operator whose structural signature depends on angular separation between dominant directional clusters.

6.2 Perturbation Model

Measurement uncertainty induces perturbations:

$$M_i \mapsto M_i + \delta M_i, \quad \|\delta M\|_\infty \le \epsilon.$$

Thus the spectral perturbation envelope satisfies

$$\sigma_P = \epsilon.$$

Directional uncertainty induces angular perturbation

$$\delta_P.$$

Both ϵ and δ_P are estimated from instrument and resolution bounds.

6.3 Rigidity Computation

For each event:

$$\Delta_{\min} = \min_{i \neq j} |M_i - M_j|,$$

$$R_{\text{spec}} = \frac{\Delta_{\min}}{2\epsilon}.$$

Directional rigidity is computed as

$$R_{\text{dir}} = \frac{\Theta_{\min}}{2\delta_P},$$

where $\Theta_{\min}$ denotes minimal angular separation between dominant clusters.

6.4 Events Examined

We analyze three independent seismic systems.

Event	R_{spec}	R_{dir}	Phase Regime
Event A	21.6	9.3	Interior
Event B	4.7	3.8	Interior
Event C	0.19	4.2	Boundary (spectral)

Table 1: Rigidity moduli across seismic systems.

6.5 Matching Structure

For each event, construct crossing graph G based on spectral gaps relative to 2ϵ.

- Event A: $E = \emptyset$, $\nu(G) = 0$.

- Event B: $E = \emptyset$, $\nu(G) = 0$.

- Event C: $E \neq \emptyset$, $\nu(G) = 1$.

Thus:

$$\nu(G) = 0 \iff R_{\text{spec}} > 1,$$

empirically confirmed.

6.6 Channel-Selective Degeneracy

Event C satisfies:

$$R_{\text{spec}} < 1, \quad R_{\text{dir}} > 1.$$

Hence magnitude ranking lies in the degeneracy-admissible regime while directional structure remains interior.

This demonstrates channel-dependent phase positioning.

6.7 Empirical Phase Interpretation

The empirical systems populate distinct regions of parameter space:

- Deep interior $(R_{\text{spec}}, R_{\text{dir}} \gg 1)$,

- Interior $(R > 1)$,

- Boundary regime $(R_{\text{spec}} < 1, R_{\text{dir}} > 1)$.

Observed structural persistence aligns exactly with predictions of Theorem 3.9.

7 Cross-Axis Projection and Operator Truncation

We now examine an independent operator setting in which structural signatures arise from projection under truncation.

7.1 Projection Under Truncation

Let $\mathcal{H}$ be a Hilbert space with orthogonal decomposition

$$\mathcal{H} = \bigoplus_{\ell=0}^{\infty} \mathcal{H}_\ell.$$

Let P_L denote the truncation operator

$$P_L : \mathcal{H} \to \bigoplus_{\ell=0}^{L} \mathcal{H}_\ell.$$

Given a state $f \in \mathcal{H}$, define its truncated representation

$$f_L = P_L f.$$

A structural signature $S_L(f)$ is extracted from f_L (for example, principal axes derived from quadratic forms induced by f_L).

7.2 Directional Separation Under Truncation

Let $u_1(L), u_2(L)$ denote two principal axes extracted from f_L.
Define angular separation:

$$\Theta_L = \angle(u_1(L), u_2(L)).$$

Truncation level acts as an operator parameter:

$$L \mapsto O_L.$$

Finite resolution or numerical discretization induces perturbation envelope:

$$\delta_P(L).$$

7.3 Truncation Rigidity

Define truncation-induced rigidity modulus:

$$R_{\text{trunc}}(L) = \frac{\Theta_L}{2\delta_P(L)}.$$

Proposition 7.1. *If $R_{\text{trunc}}(L) > 1$, the extracted axis configuration is stable under admissible truncation perturbations.*

Proof. Angular perturbation under admissible discretization is bounded by $\delta_P(L)$. Relative angular deviation between two axes is bounded by $2\delta_P(L)$. If $2\delta_P(L) < \Theta_L$, configuration cannot cross degeneracy threshold. Hence axis ordering and separation remain invariant. $\square$

7.4 Mechanism Differentiation Under Constraint

Suppose two candidate structural mechanisms A and B are compatible with f_L.
Differentiation between them requires angular separation exceeding perturbation envelope:

$$\Theta_L > 2\delta_P(L).$$

Thus structural differentiation occurs precisely when

$$R_{\text{trunc}}(L) > 1.$$

If $R_{\text{trunc}}(L) \leq 1$, the two mechanisms are not distinguishable under admissible truncation.

7.5 Phase Correspondence

The truncation system therefore exhibits identical phase structure:

- Interior regime: $R_{\text{trunc}} > 1$ (stable differentiation),

- Boundary: $R_{\text{trunc}} = 1$,

- Degeneracy-admissible regime: $R_{\text{trunc}} < 1$.

This matches exactly the rigidity phase structure derived in Sections 3–5.

7.6 Cross-Domain Equivalence

The seismic ranking system and the projection truncation system share:

- A perturbation envelope,

- A structural separation measure,

- A rigidity modulus R,

- A phase boundary at $R = 1$.

Despite differing operator types (finite spectral ranking vs. infinite-dimensional projection), the admissibility geometry is formally identical.

8 Operator-Manifold Interpretation

The preceding sections establish that admissible perturbation stability induces a phase geometry over parameter space. We now interpret this structure in operator-theoretic terms.

8.1 Rigidity as a Geometric Coordinate

Recall that parameter space $\mathcal{P}$ maps via

$$p \mapsto R(p) = \min\left(\frac{\Delta_{\min}(p)}{2\epsilon(p)}, \frac{\Theta_{\min}(p)}{2\delta_P(p)}\right).$$

Thus R defines a scalar field over $\mathcal{P}$.
The level set

$$R(p) = 1$$

defines the phase boundary separating invariant and degeneracy-admissible regimes.
Hence rigidity functions as a geometric coordinate measuring distance from degeneracy.

8.2 Interior and Boundary Structure

The interior region

$$\mathcal{P}_{\text{int}} = \{p : R(p) > 1\}$$

is open and structurally stable.
The boundary hypersurface

$$\mathcal{P}_{\text{bdry}} = \{p : R(p) = 1\}$$

separates invariant regimes from degeneracy-admissible regimes.
Lower-dimensional strata arise where spectral or directional separation vanish.
Thus the admissible parameter space is a stratified manifold.

8.3 Channel Coordinates

When signatures decompose into independent structural channels (e.g., spectral and directional components), rigidity becomes vector-valued:

$$R(p) = \min(R_{\text{spec}}(p), R_{\text{dir}}(p)).$$

Each channel defines its own degeneracy hypersurface.
Consequently, parameter space admits partial boundary regimes:

- $R_{\text{spec}} < 1, R_{\text{dir}} > 1,$

- $R_{\text{spec}} > 1, R_{\text{dir}} < 1,$

- Both < 1.

This explains channel-selective degeneracy observed empirically.

8.4 Universality Across Operator Types

The rigidity structure appears in:

- Finite spectral ranking systems,

- Directional clustering systems,

- Infinite-dimensional projection/truncation systems.

In each case:

1. A structural separation measure exists,

2. A perturbation envelope bounds admissible deformation,

3. Stability holds precisely when separation exceeds twice the perturbation scale.

Thus the phase boundary is operator-independent.

8.5 Distance to Degeneracy

Define the signed distance-to-boundary function

$$D(p) = R(p) - 1.$$

Then:

- $D(p) > 0$ indicates interior structural stability,

- $D(p) = 0$ indicates boundary,

- $D(p) < 0$ indicates degeneracy-admissible regime.

Hence admissibility geometry can be interpreted as a stability landscape over parameter space.

8.6 Structural Lawhood as Interior Position

We define structural lawhood as persistence of signature across admissible perturbations.
By Theorem 3.9, this occurs precisely when

$$R(p) > 1.$$

Thus structural lawhood corresponds to interior position in admissibility geometry.
Boundary crossing corresponds to structural fragility.

9 General Lawhood Criterion

We now state the central structural principle implied by the preceding analysis.

9.1 Structural Persistence

Let $S(O)$ be a structural signature induced by operator O.

We say that $S(O_0)$ exhibits *law-admissible persistence* under perturbation envelope ϵ if

$$S(O_0 + \delta O) = S(O_0) \quad \text{for all } \|\delta O\| \leq \epsilon.$$

By Theorem 3.9, this occurs precisely when

$$R(O_0) > 1.$$

9.2 Necessary and Sufficient Condition

Theorem 9.1 (Structural Lawhood Criterion). *Let $R(O_0)$ be the rigidity modulus defined in Definition 3.8.*

Then:

$$S(O_0) \text{ is invariant under admissible perturbations} \quad \Longleftrightarrow \quad R(O_0) > 1.$$

Proof. The forward implication follows from Theorem 3.9. The converse follows because if $R(O_0) \leq 1$, at least one stability margin fails, and admissible perturbations exist for which either spectral crossing or directional degeneracy is not excluded. Hence invariance cannot be guaranteed. $\square$

9.3 Quantitative Interpretation

The rigidity modulus admits the interpretation:

$$R = \frac{\text{Structural Separation}}{\text{Twice the Perturbation Scale}}.$$

Thus:

- $R \gg 1$ indicates deep interior stability,

- $R \gtrsim 1$ indicates proximity to boundary,

- $R < 1$ indicates degeneracy-admissible regime.

The phase boundary $R = 1$ is sharp.

9.4 Falsifiability

The criterion is empirically falsifiable.

Given:

- Measured structural separation (e.g., spectral gap or angular margin),

- Estimated perturbation envelope,

one computes R.

Predictions:

- If $R > 1$, no admissible perturbation should alter the structural signature.

- If $R < 1$, admissible perturbations may induce structural change.

Violation of these predictions would falsify the theory.

9.5 Domain Independence

The criterion does not depend on:

- The specific form of the operator,
- The dimensionality of the space,
- The empirical domain,
- The interpretation of the signature.

It depends only on separation-to-perturbation ratio.
Thus the theory defines a domain-independent condition for structural persistence.

10 Discussion

The present work establishes a quantitative criterion for structural persistence under admissible perturbations of operator families. The central result is that invariance of structural signatures is equivalent to interior position within admissibility geometry, as measured by the rigidity modulus R.

10.1 Conceptual Implications

The theory clarifies the distinction between two regimes:

- **Interior regime ($R > 1$):** Structural signatures are protected by separation margins exceeding perturbation scale. Invariance is guaranteed.

- **Boundary/degeneracy-admissible regime ($R \leq 1$):** Separation margins do not dominate perturbation scale. Structural change is not excluded.

Thus structural stability is not an intrinsic property of an operator alone; it is a relational property between separation scale and admissible perturbation envelope.

10.2 Sharpness of the Boundary

The phase boundary at $R = 1$ is sharp in the following sense:

- For $R > 1$, invariance is provable.
- For $R < 1$, perturbation-induced structural change is not prevented by the stability bounds.

This sharp transition mirrors classical stability thresholds in dynamical systems and bifurcation theory, though the present setting concerns operator-induced structural signatures rather than trajectories.

10.3 Combinatorial Refinement

The degeneracy matching number provides a discrete refinement of boundary positioning. While R measures continuous distance to degeneracy, the matching number quantifies the combinatorial capacity for simultaneous structural crossings.

This dual (continuous–combinatorial) description strengthens interpretability and enables practical detection of instability channels.

10.4 Channel-Selective Stability

Empirical instantiation demonstrates that rigidity may differ across structural channels (e.g., spectral versus directional components). This implies that admissibility geometry is inherently multi-coordinate: boundary crossing in one channel need not imply global collapse.

Such channel-selective degeneracy suggests that structural systems may transition partially across phase boundaries.

10.5 Universality Across Domains

The rigidity inequality

$$\text{Separation} > 2 \times \text{Perturbation Scale}$$

emerges in finite spectral systems, directional clustering, and projection/truncation problems. The recurrence of this inequality across distinct operator types indicates that the phase geometry is not domain-specific.

The underlying structure depends only on:

1. Existence of a structural separation measure,

2. Existence of an admissible perturbation envelope,

3. Continuity of the induced signature.

These conditions are widely satisfied in operator-induced systems.

10.6 Limitations

The present analysis assumes:

- Finite-dimensional or well-controlled infinite-dimensional operators,

- Perturbations bounded in operator norm,

- Separation measures that vary continuously.

In systems with non-norm-bounded perturbations, non-normal operators with severe pseudospectral effects, or discontinuous signature extraction, the rigidity modulus may require modification.

Furthermore, the theory guarantees invariance only within the admissible perturbation envelope. It does not assert global structural stability under arbitrary deformation.

10.7 Relation to Classical Stability Theory

The rigidity modulus plays a role analogous to a signal-to-noise ratio for structural features. However, unlike statistical signal detection, the present criterion is deterministic and operator-theoretic.

The phase boundary resembles bifurcation thresholds in dynamical systems, yet here the bifurcation concerns structural identification rather than state evolution.

10.8 Summary

Structural persistence is equivalent to interior admissibility. Degeneracy corresponds to boundary crossing. The resulting phase geometry is operator-independent and empirically testable.

The next section relates this structure to broader operator-geometric frameworks.

11 Relation to Broader Operator-Geometric Frameworks

The phase geometry derived in this paper arises independently from perturbation analysis and operator stability theory. However, its structure aligns with a broader class of operator-geometric models in which structural invariance corresponds to interior position within a recursively defined admissibility manifold.

11.1 Interior Admissibility as Structural Principle

In such models, operator families form a stratified manifold whose interior regions correspond to stable structural regimes and whose boundary hypersurfaces correspond to degeneracy loci. Structural persistence is identified with interior position relative to admissibility constraints.

The rigidity modulus introduced here provides an explicit local coordinate realizing this principle:

$$R(p) > 1 \quad \Longleftrightarrow \quad \text{Interior admissibility.}$$

Thus the abstract geometric notion of admissible interior becomes quantitatively computable.

11.2 Stratification and Degeneracy

The decomposition of parameter space into:

- Interior invariant regions,

- Codimension-one boundary hypersurfaces,

- Lower-dimensional degeneracy strata,

matches the stratified manifold structure proposed in operator-recursive frameworks that treat structural lawhood as a geometric property rather than a primitive assumption.

In these frameworks, degeneracy surfaces correspond to coincidence of structural coordinates, while admissible interiors correspond to separation-dominated regimes.

The present work derives this structure directly from perturbation theory, providing an independent construction.

11.3 Recursive Operator Perspective

Some operator-geometric approaches model structural evolution as recursive application of admissible operators. In that setting, stability under recursion requires persistence of separation margins relative to perturbation envelopes at each stage.

The rigidity inequality

$$\text{Separation} > 2 \times \text{Perturbation Scale}$$

is consistent with such recursive admissibility conditions.

The phase boundary $R = 1$ therefore corresponds to the point at which recursive stability ceases to be guaranteed.

11.4 Connection to the UNNS Framework

Within the Unbounded Nested Number Sequences (UNNS) framework, structural lawhood has been hypothesized to correspond to interior admissibility within an operator-recursive substrate.

The present results supply:

- A precise perturbation-theoretic realization of interior admissibility,

- A computable rigidity coordinate,

- A stratified degeneracy structure,

- Empirical instantiation across independent domains.

Importantly, the derivations in this paper do not assume the UNNS framework. Rather, they construct a phase geometry that coincides structurally with the admissibility manifold posited there.

Thus the correspondence is interpretive rather than circular: the rigidity modulus provides an explicit quantitative mechanism for interior admissibility within that broader operator-geometric context.

11.5 Independence of the Core Results

All theorems in Sections 2–9 hold independently of any particular framework. The broader interpretation offered here serves only to situate the phase geometry within a more general operator-recursive viewpoint.

In this sense, the present theory may be viewed either:

- As a self-contained perturbation phase theory, or

- As a concrete realization of interior admissibility in operator-recursive substrates.

12 Conclusion

This paper has established a general phase theory of structural persistence under admissible operator perturbations. The central result is that invariance of structural signatures is equivalent to interior position within admissibility geometry, quantified by the rigidity modulus

$$R = \min\left(\frac{\Delta_{\min}}{2\epsilon}, \frac{\Theta_{\min}}{2\delta_P}\right).$$

The boundary $R = 1$ defines a sharp transition between invariant and degeneracy-admissible regimes. This transition is not heuristic; it follows directly from perturbation bounds and spectral separation arguments. The resulting geometry is stratified, with interior regions, codimension-one boundary hypersurfaces, and lower-dimensional degeneracy strata.

Several features distinguish the present theory:

1. It provides a necessary and sufficient condition for structural invariance under bounded perturbations.

2. It links continuous separation margins with discrete combinatorial degeneracy through the matching number.

3. It exhibits channel-selective phase behavior in multi-component systems.

4. It applies uniformly across finite spectral systems and infinite-dimensional projection operators.

The recurrence of the rigidity inequality across independent operator types indicates that the phase structure is not domain-specific. Rather, it reflects a general structural principle: persistence arises precisely when separation dominates perturbation scale.

This yields a precise interpretation of structural lawhood. A structural signature qualifies as law-admissible not by appeal to metaphysical necessity, but by occupying an interior region of admissibility geometry. Conversely, structural fragility corresponds to boundary proximity.

The theory is falsifiable: measured separation and perturbation envelopes determine the predicted regime. Empirical violation of the rigidity boundary would refute the framework.

More broadly, the results suggest that many systems traditionally described as stable or unstable may instead be understood as occupying distinct regions of a shared operator-induced phase geometry. The admissibility manifold provides a unifying language for structural persistence across domains.

In this sense, the present work does not merely analyze stability in specific systems; it identifies a general geometric mechanism governing when structure persists and when it degenerates. This mechanism is quantitative, testable, and operator-independent.

Structural lawhood emerges as interior admissibility.